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A Note on Littlewood–Paley Decompositions with Arbitrary Intervals

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DEDICATED TO THE MEMORY OF GÉZA FREUD

If I is an interval in $\mathbb{R}$ we define an operator S_I by setting $(S_I f)^{\wedge} = \chi_I \hat{f}$, where $\hat{f}$ denotes the Fourier transform of f. Let $(I_k)_1^{\infty}$ be a sequence of disjoint intervals and set

$$\Delta f(x) = \left(\sum_1^{\infty} |S_{I_k} f(x)|^2 \right)^{1/2}.$$

Rubio de Francia [2] has proved the following theorem.

THEOREM A. *Assume that $2 \leqslant p < \infty$. Then there exists a constant C_p such that*

$$\|\Delta f\|_p \leqslant C_p \|f\|_p, \qquad f \in L^p(\mathbb{R}). \tag{1}$$

It is also proved that C_p may be chosen independently of the sequence $(I_k)_1^{\infty}$. We shall give here an alternative proof of the basic inequality in the proof of (1).

A reduction argument in [2] shows that in proving the theorem one may assume that

$$\sum_1^{\infty} \chi_{8I_k}(x) \leqslant C. \tag{2}$$

We let $(I_k)_1^{\infty}$ be a sequence satisfying (2) and let $(I_k^j)_j$ denote the intervals I in the sequence which satisfy $2^k \leqslant |I| < 2^{k+1}$, $k \in \mathbb{Z}$. Choose integers n_k^j such that $n_k^j 2^k \in I_k^j$. Then choose a function φ in the Schwartz class $\mathscr{S}$ such that $\hat{\varphi}(\xi) = 1$, $|\xi| \leqslant 2$, and $\hat{\varphi}(\xi) = 0$, $|\xi| \geqslant 3$, where

$$\hat{\varphi}(\xi) = \int e^{-i2\pi\xi x} \varphi(x)\, dx.$$

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We then set $\varphi_k^j(x) = 2^k \varphi(2^k x)\, e^{i2\pi 2^k n_k^j x}$ so that

$$(\varphi_k^j)\hat{\ }(\xi) = \hat{\varphi}(2^{-k}\xi - n_k^j) = \begin{cases} 1, & \xi \in I_k^j \\ 0, & \xi \notin 8I_k^j. \end{cases}$$

We define operators G and M_q by setting

$$Gf(x) = \left(\sum_{k,j} |\varphi_k^j * f(x)|^2 \right)^{1/2}$$

and

$$M_q f(x) = [M(|f|^q)(x)]^{1/q}, \qquad 1 \leqslant q < \infty,$$

where M denotes the Hardy–Littlewood maximal operator. The main step in the proof in [2] is the inequality

$$(Gf)^{\#}(x) \leqslant CM_2 f(x), \tag{3}$$

which is proved for every bounded function f with compact support. Here $(Gf)^{\#}$ denotes the sharp maximal function of Gf (see Fefferman and Stein [1]).

We shall give here an alternative proof of (3). Let I_0 denote an interval and assume that $x_0 \in I_0$. We have to prove that

$$\frac{1}{|I_0|} \int_{I_0} |Gf(x) - (Gf)_{I_0}|\, dx \leqslant CM_2 f(x_0), \tag{4}$$

where we use the notation

$$g_I = \frac{1}{|I|} \int_I g\, dx.$$

It is clearly sufficient to prove that there exists a constant $a = a(x_0, I_0, f)$ such that

$$\frac{1}{|I_0|} \int_{I_0} |Gf(x) - a|\, dx \leqslant CM_2 f(x_0). \tag{5}$$

We now fix x_0 and I_0 and prove (5). First let k_0 be determined by the inequality $2^{-k_0-1} < |I_0| \leqslant 2^{-k_0}$. Then write $f = g + h$, where $g = f\chi_{2I_0}$. We shall prove that (5) holds with

$$a = \left(\sum_{k \leqslant k_0 + 2} |\varphi_k^j * h(x_0)|^2 \right)^{1/2}.$$

Using the triangle inequality in l^2, we obtain

$$|Gf(x)-a| \leqslant |Gf(x)-Gh(x)| + |Gh(x)-a| \leqslant Gg(x) + \left(\sum_{k \geqslant k_0+3} |\varphi_k^j * h(x)|^2\right)^{1/2} + \left(\sum_{k \leqslant k_0+2} |F_k^j(x)|^2\right)^{1/2} = A(x)+B(x)+C(x),$$

where

$$F_k^j(x) = \int 2^k(\varphi(2^k x - 2^k y) - \varphi(2^k x_0 - 2^k y))\, e^{-i2\pi 2^k n_k^j y} h(y)\, dy.$$

Invoking the Plancherel theorem we conclude that

$$\frac{1}{|I_0|}\int_{I_0} A(x)\, dx \leqslant \frac{1}{|I_0|}\left(\int_{I_0} A^2\, dx\right)^{1/2} |I_0|^{1/2} \leqslant \frac{1}{|I_0|^{1/2}}\, C\left(\int |g|^2\, dx\right)^{1/2} = C\left(\frac{1}{|I_0|}\int_{2I_0} |f|^2\, dx\right)^{1/2} \leqslant CM_2 f(x_0), \tag{6}$$

since $\sum |(\varphi_k^j)\hat{}\,|^2 \leqslant C$.

We shall now estimate $B(x)$. We first set $J_k^l = [l2^{-k-1},\ l2^{-k-1} + 2^{-k}]$ for $l \in \mathbb{Z}$, $k \in \mathbb{Z}$. Then choose $\psi_k^l \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \psi_k^l$ is contained in the interior of J^l, $\sum_l \psi_k^l = 1$, and

$$|D^m \psi_k^l| \leqslant C_m 2^{km}, \qquad m = 0, 1, 2, \ldots .$$

We now assume that $k \geqslant k_0 + 3$ and $x \in I_0$. We have

$$|\varphi_k^j * h(x)| = \left|\int 2^k \varphi(2^k x - 2^k y)\, e^{-i2\pi 2^k n_k^j y} h(y)\, dy\right| = \left|\sum_l \int_{J_k^l} 2^k \varphi(2^k x - 2^k y)\, e^{-i2\pi 2^k n_k^j y} \psi_k^l(y)\, h(y)\, dy\right|.$$

Expansion in Fourier series yields

$$\varphi(2^k x - 2^k y)\, \psi_k^l(y) = \sum_n a_n(x, l)\, e^{i2\pi 2^k n y}, \qquad y \in J_k^l,$$

where

$$a_n(x, l) = 2^k \int_{J_k^l} \varphi(2^k x - 2^k y)\, \psi_k^l(y)\, e^{-i2\pi 2^k n y}\, dy.$$

It follows that

$$|\varphi_k^j * h(x)| \leqslant \sum_l \sum_n |a_n(x, l)|\, |c_{n_k^j - n}(h, J_k^l)|,$$

where

$$c_n(h, J_k^l) = 2^k \int_{J_k^l} e^{-i2\pi 2^k n y} h(y)\, dy.$$

If $J_k^l \subset 2I_0$ then h vanishes on J_k^l and if $J_k^l \not\subset 2I_0$ we integrate by parts and obtain

$$\begin{aligned} |a_n(x, l)| &\leqslant 2^k \frac{1}{2^{2k} n^2} \int_{J_k^l} |D_y^2[\varphi(2^k x - 2^k y)\, \psi_k^l(y)]|\, dy \\ &\leqslant C \frac{2^k}{n^2} \int_{J_k^l} \frac{1}{2^{2k}|x-y|^2}\, dy \leqslant C \frac{1}{n^2 2^{2k} d(x_0, J_k^l)^2}, \qquad n \neq 0, \quad x \in I_0, \end{aligned}$$

since $|x-y| \geqslant 2^{-k_0-3}$ for $y \in J_k^l$. Here $d(x_0, J)$ denotes the distance between x_0 and J.

A similar estimate holds for $n = 0$ and we therefore have

$$|a_n(x, l)| \leqslant C \frac{1}{(1+n^2)\, 2^{2k}\, d(x_0, J_k^l)^2}, \qquad n \in \mathbb{Z}, \quad x \in I_0,$$

for $J_k^l \not\subset 2I_0$. We now set

$$C_m(h, J_k^l) = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} |c_{m+n}(h, J_k^l)|, \qquad m \in \mathbb{Z},$$

and it follows that

$$|\varphi_k^j * h(x)| \leqslant C \sum_l \frac{1}{2^{2k} d(x_0, J_k^l)^2} C_{n_k^j}(h, J_k^l).$$

An application of the Schwarz inequality shows that

$$|\varphi_k^j * h(x)|^2 \leqslant C \sum_l \frac{1}{2^{2k} d(x_0, J_k^l)^2} C_{n_k^j}(h, J_k^l)^2.$$

We shall now use the estimate

$$\sum_m C_m(h, J_k^l)^2 \leqslant C 2^k \int_{J_k^l} |h|^2\, dy,$$

which is a consequence of the Parseval relation. We obtain

$$\begin{aligned}\sum_j |\varphi_k^j * h(x)|^2 &\leqslant C \sum_l \frac{1}{2^{2k} d(x_0, J_k^l)^2} \sum_j C_{n_k^j}(h, J_k^l)^2 \\ &\leqslant C \sum_l \frac{1}{2^{2k} d(x_0, J_k^l)^2} 2^k \int_{J_k^l} |h|^2 \, dy \\ &\leqslant C \sum_l \int_{J_k^l} \frac{2^k}{2^{2k}|x_0 - y|^2} |h(y)|^2 \, dy \\ &\leqslant C \int \frac{2^k}{2^{2k}|x_0 - y|^2} |h(y)|^2 \, dy.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{\substack{j \\ k \geqslant k_0+3}} |\varphi_k^j * h(x)|^2 &\leqslant C \int \frac{2^{k_0}}{2^{2k_0}|x_0 - y|^2} |h(y)|^2 \, dy \\ &\leqslant C \int 2^{k_0} \psi(2^{k_0}(x_0 - y)) |h(y)|^2 \, dy \leqslant CM(|h|^2)(x_0) \\ &\leqslant CM(|f|^2)(x_0), \qquad x \in I_0,\end{aligned}$$

where

$$\psi(y) = \frac{1}{1 + y^2}.$$

We conclude that

$$B(x) \leqslant CM_2 f(x_0), \qquad x \in I_0. \tag{7}$$

It remains to estimate $C(x)$ and we therefore assume $k \leqslant k_0 + 2$. We have

$$F_k^j(x) = \sum_l \int_{J_k^l} 2^k(\varphi(2^k x - 2^k y) - \varphi(2^k x_0 - 2^k y)) \, \psi_k^l(y) \, e^{-i2\pi 2^k n_k^j y} h(y) \, d(y)$$

and arguing as above we obtain

$$|F_k^j(x)| \leqslant \sum_l \sum_n |b_n(x, l)| \, |c_{n_k^j - n}(h, J_k^l)|,$$

where

$$b_n(x, l) = 2^k \int_{J_k^l} [\varphi(2^k x - 2^k y) - \varphi(2^k x_0 - 2^k y)] \, \psi_k^l(y) \, e^{-i2\pi 2^k n y} \, dy.$$

Integrating by parts twice and using the mean value theorem we obtain

$$|b_n(x, l)| \leqslant 2^k \frac{1}{2^{2k}n^2} \int_{J_k^l} |D_y^2\{[\varphi(2^k x - 2^k y) - \varphi(2^k x_0 - 2^k y)]\, \psi_k^l(y)\}|\, dy$$

$$\leqslant C 2^{2k} \frac{|x - x_0|}{n^2} \int_{J_k^l} \frac{1}{1 + 2^{2k}|x_0 - y|^2}\, dy$$

$$\leqslant C 2^{k - k_0} \frac{1}{n^2(1 + 2^{2k}\, d(x_0, J_k^l)^2)}$$

for $n \neq 0$, $x \in I_0$.

An analogous estimate holds for $n = 0$ and we conclude that

$$|b_n(x, l)| \leqslant C 2^{k - k_0} \frac{1}{(1 + n^2)(1 + 2^{2k}\, d(x_0, J_k^l)^2)}, \qquad n \in \mathbb{Z},\ x \in I_0.$$

It follows that

$$|F_k^j(x)| \leqslant C 2^{k - k_0} \sum_l \frac{1}{1 + 2^{2k}\, d(x_0, J_k^l)^2}\, C_{n_k^j}(h, J_k^l)$$

and an application of the Schwarz inequality shows that

$$|F_k^j(x)|^2 \leqslant C 2^{2(k - k_0)} \sum_l \frac{1}{1 + 2^{2k}\, d(x_0, J_k^l)^2}\, C_{n_k^j}(h, J_k^l)^2.$$

Summing over j we obtain

$$\sum_j |F_k^j(x)|^2 \leqslant C 2^{2(k - k_0)} \sum_l \frac{1}{1 + 2^{2k} d(x_0, J_k^l)^2}\, 2^k \int_{J_k^l} |h|^2\, dy$$

$$\leqslant C 2^{2(k - k_0)} \int \frac{2^k}{1 + 2^{2k}|x_0 - y|^2}\, |h(y)|^2\, dy$$

$$= C 2^{2(k - k_0)} \int 2^k \psi(2^k(x_0 - y))|h(y)|^2\, dy$$

$$\leqslant C 2^{2(k - k_0)} M(|f|^2)(x_0).$$

Hence,

$$\sum_{\substack{j \\ k \leqslant k_0 + 2}} |F_k^j(x)|^2 \leqslant C M(|f|^2)(x_0), \qquad x \in I_0,$$

and

$$C(x) \leqslant C M_2 f(x_0), \qquad x \in I_0. \tag{8}$$

Inequality (5) follows from the estimates (6), (7), and (8) and thus the proof of (3) is complete.

References

1. C. Fefferman and E. M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193.
2. J. L. Rubio de Francia, "A Littlewood–Paley inequality for arbitrary intervals," Report No. 18, Institut Mittag-Leffler, 1983.

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